Sunday, November 29, 2009
7:58 PM

This can be called request rate, or customer arrival rate.

Poisson Process (2)

Last time, we have seen that we can approximate a Poisson process (PP) with an arrival rate λ by a Bernoulli process which divides the time axis into slots of time duration $\frac{I}{n}$ each. Then, for all slots, we model the arrival process as i.i.d. Bernoulli random variables with probability of being 1 equal to $\lambda \frac{I}{n}$.

We have assumed that this approximation would become accurate when the slot duration ($\frac{I}{n}$) is small.

One important property of PP that we discussed last time is that

> if we consider non-overlapping intervals, then the number of arrivals in these intervals are independent random variables.

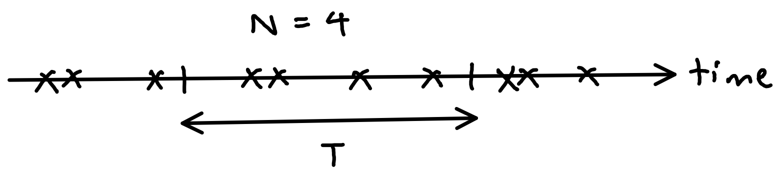
This is the reason why we have the first "i" in the "i.i.d." Bernoulli random variables above. This "i" stands for "independent."

Another "i" in "i.i.d." stands for "identically" which indicates that all independent Bernoulli r.v.'s associated with the slots share the same pmf. For Bernoulli r.v., pmf is defined easily by the probability of being 1. In this case, it is then clear why all of them have the same pmf; the probability of being 1 is $\lambda \frac{I}{n}$ for all slots.

What we want to do next is to reconsider the number of arrivals during a time interval. However, we will <u>not</u> assume that the interval is small. In which case, we can not assume that there is at most one arrival in this interval of interest. The number of arrivals, which will be denoted by N, can be any nonnegative integer (0, 1, 2, 3, ...). We would like to find the exact pmf of N.

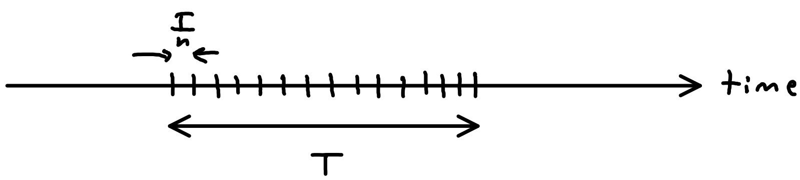
To do this, we will use the same approach that we have done last time. Eventhough our interval can be large, we can always divide it into small intervals/slots. In which case, the Bernoulli assumption is valid again.

Let's consider an interval of length T below



Let N be the number of arrivals during this time interval. In the figure above, $N = 4$.

We divide T into n small intervals.

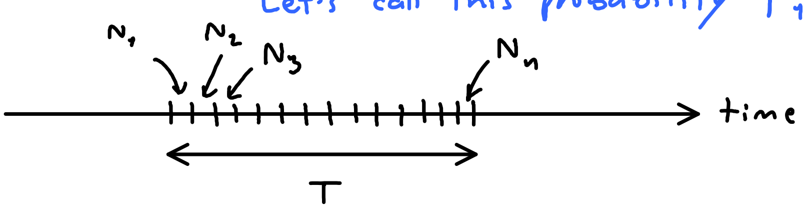


Each of the small interval then has length $\frac{T}{n}$.

From last time, we know that the number of arrivals in these intervals, denoted by $N_1, N_2, N_3, N_4, \dots, N_n$ as shown below will be

$$\text{i.i.d. Bernoulli with probability of being 1} = \lambda \frac{T}{n}$$

Let's call this probability p_1



The total number of arrivals during the original interval of length T is then

$$N \approx N_1 + N_2 + N_3 + \dots + N_n.$$

Of course, this is an approximation. Again, it is

Of course, this is an approximation. Again, it is accurate when $\frac{I}{n}$ is small which will happen when n is large.

So, we expect that

$$\lim_{n\to\infty} (N_1 + N_2 + \cdots + N_n)$$

would be the same as N.

However, what is

$$\lim_{n\to\infty} N_1 + N_2 + \cdots + N_n \quad ?!$$

First let's define

$$S_n = N_1 + N_2 + \cdots + N_n$$

You may recall, from your probability class, that S_n is a Binomial r.v.

and study S_n more carefully.

For each N_k, where $k = 1, 2, \ldots, n$

N_k is a Bernoulli random variable with $p_1 = \mathbb{E} N_k = \lambda \frac{I}{n}$.

When studying a sum of independent r.v.'s as in S_n above, it is traditional to use characteristic funtion.

Recall that for a random variable X, the characteristic function is given by

$$\varphi_X(u) = \mathbb{E}\, e^{jXu}$$

$$= \sum_x P[X = x]\, e^{jxu} \quad \text{for discrete r.v.}$$

Note how similar this is to a Fourier transform. →

$$= \int f_X(x)\, e^{jxu}\, dx \quad \text{for continuous r.v.}$$

For Bernoulli r.v. N_k,

$$\varphi_{N_k}(u) = P[N_k = 0]\, e^{j0u} + P[N_k = 1]\, e^{j1u}$$

$$= P[N_k = 0] + P[N_k = 1]\, e^{ju}$$

$$= \left(1 - \frac{\lambda T}{n}\right) + \frac{\lambda T}{n} e^{ju}.$$

Now, when we have a sum of two independent r.v. X and Y, recall that the sum pdf or pmf is a convolution of the pdf's or pmf's of X and Y:

$$p_{X+Y}(z) = \sum_{x} p_Y(z-x)\, p_X(x) \quad \text{for discrete r.v.}$$

$$f_{X+Y}(z) = \int f_Y(z-x)\, f_X(x)\, dx \quad \text{for continuous r.v.}$$

Therefore, the characteristic function of the sum is a product of the characteristic functions of X and Y:

(Recall that convolution in time domain is a product in frequency domain)

$$\varphi_{X+Y}(u) = \varphi_X(u)\, \varphi_Y(u).$$

Using this property of characteristic function, we then have

$$\varphi_{S_n}(u) = \varphi_{N_1}(u) \times \cdots \times \varphi_{N_n}(u)$$

$$= \left(1 - \frac{\lambda T}{n} + \frac{\lambda T}{n} e^{ju}\right)^n$$

$$= \left(1 + \frac{1}{n}\left(-\lambda T + \lambda T e^{ju}\right)\right)^n.$$

Recall, from calculus class, that

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \to e^{x}$$

Hence, $\varphi_{S_n}(u) \to \exp\left(-\lambda T + \lambda T e^{ju}\right)$

for any u

as $n \to \infty$.

This implies that the characteristic function of N (the number of arrivals in interval of length T) is

$$\varphi_N(u) = \exp\left(-\lambda T + \lambda T e^{ju}\right).$$

This is the characteristic function of a Poisson random variable with mean λT.

To see this, note that if M is a Poisson r.v. with mean λT, then

$$P[M=k] = e^{-\lambda T}\frac{(\lambda T)^k}{k!}$$

So, $$\varphi_M(u) = \sum_{k=0}^{\infty} P[M=k]\, e^{juk}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^k}{k!} e^{juk}$$

$$= e^{-\lambda T}\sum_{k=0}^{\infty}\frac{(\lambda T e^{ju})^k}{k!}$$

Recall, from your calculus class, that $e^{x} = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$= e^{-\lambda T} e^{\lambda T e^{ju}}$$

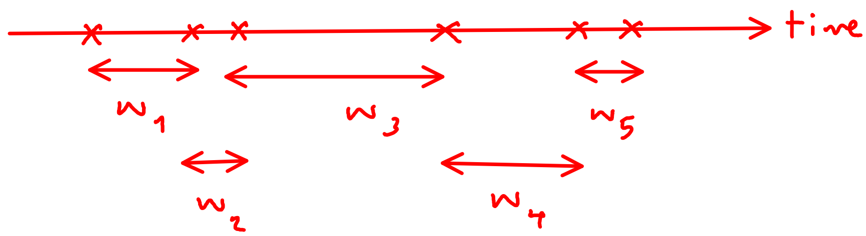
$$= \exp\left(-\lambda T + \lambda T e^{ju}\right)$$

This is exactly $\varphi_N(u)$ above.

Therefore, if N is the number of arrivals in an interval of length T, then N is a Poisson r.v. with mean λT.

There is one more fact which is widely known about PP.

The length $W_1, W_2, W_3, \ldots$ of time between adjacent arrivals are i.i.d. exponential r.v.'s with parameter λ:

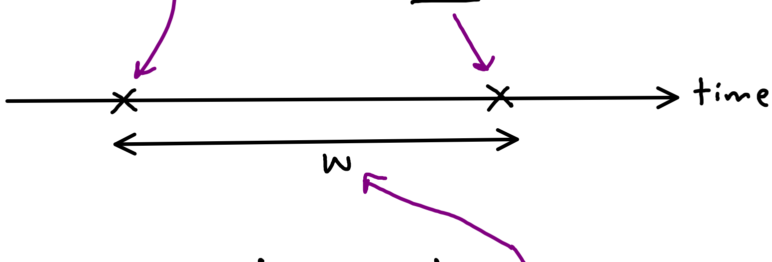


$$f_{W_k}(w) = \lambda e^{-\lambda w}, \quad w > 0.$$

Note that now we are talking about continuous random variables. We are not counting arrivals as we have done earlier. We are measuring the amount of time between arrivals.

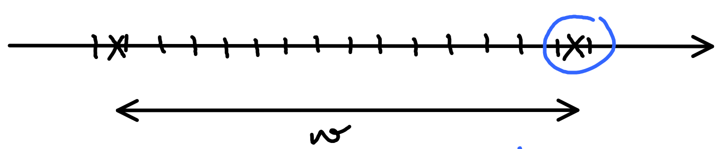
You probably know by now that we will verify this property of PP by working with small intervals!

Let consider one arrival and try to find when would be the time of the next arrival.



Denote this amount of time by W.

Now, let's work with small interval again:



We will approach the probability of having the next arrival in a slot which is about w time unit from the previous one via two methods

Method 1: Continuous :

Because the interval length is $\frac{1}{n}$, the probability of W being in this interval is $\approx f_W(w) \times \frac{T}{n}$

Method 2 : Discrete : It takes $\approx \frac{w}{T/n} = \frac{wn}{T}$ slots to find another arrival. So, during the first $\frac{wn}{T} - 1$ slots, there is no arrival.

The last one has one arrival. The probability of such event is

$$\Big(\underbrace{1 - \lambda \tfrac{T}{n}}_{1-p_1}\Big)^{\frac{wn}{T} - 1} \Big(\underbrace{\lambda \tfrac{T}{n}}_{p_1}\Big)$$

The two probability formulas approximate the same quantity and hence they should be equal:

$$f_W(w) \times \cancel{\frac{T}{n}} \approx \left(1 - \frac{\lambda T}{n}\right)^{\frac{wn}{T} - 1} \lambda \cancel{\frac{T}{n}}$$

of course, the small interval approximation won't be accurate, unless interval are extremely small. This happens when $n \to \infty$.

For the left-hand side (LHS)

$$\lim_{n\to\infty} \left(1 - \frac{\lambda T}{n}\right)^{\frac{wn}{T} - 1} \lambda$$

$$= \lim_{n\to\infty} \Big(1 - \underbrace{\cancel{\tfrac{\lambda T}{n}}}_{0}\Big)^{-1} \left(\left(1 + \frac{(-\lambda T)}{n}\right)^n\right)^{\frac{w}{T}} \lambda$$

$$= \left(e^{-\lambda T}\right)^{\frac{w}{T}} \lambda$$

$$= \lambda e^{-\lambda w}$$

we use $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ again!

So,

$$f_W(w) = \lambda e^{-\lambda w},$$

the exponential pdf as advertised above!